

# A quest for find a new map operator

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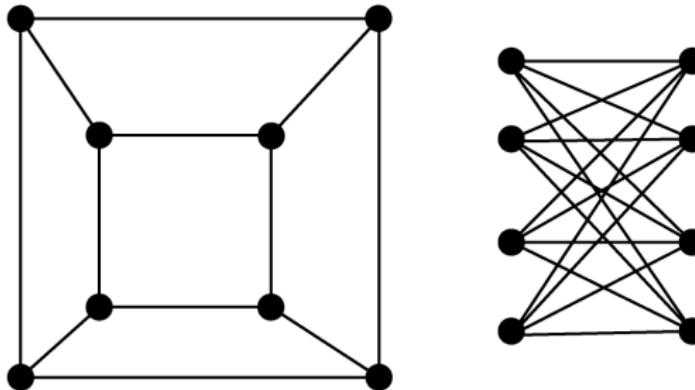
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# Graphs, Maps, Flags

A graph is an abstract discrete structure.

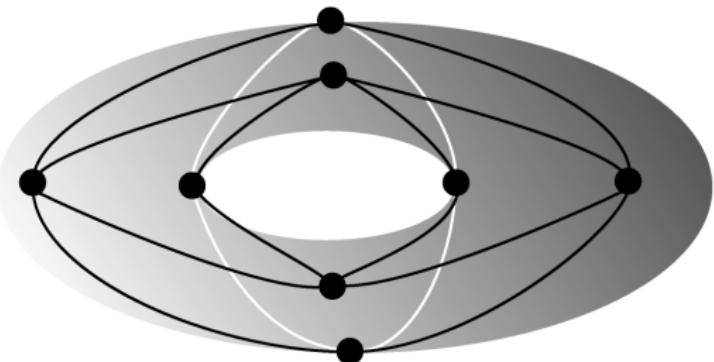
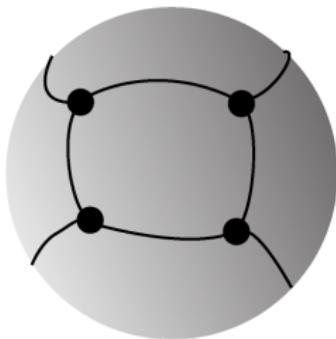
A graph consist of edges and vertices, we allow multiedges, loops, and semiedges.



# Graphs, Maps, Flags

A map is a graph drawn on a surface.

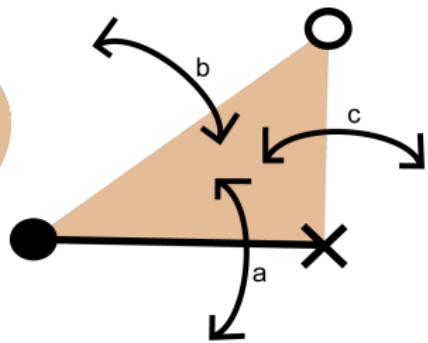
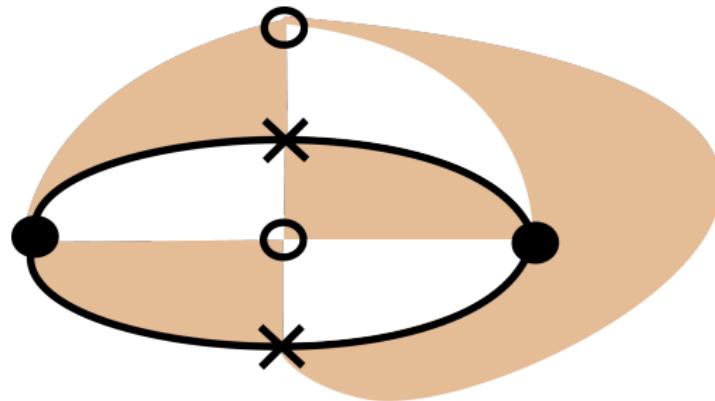
A map consists of the edges, the vertices, and a face.



# Graphs, Maps, Flags

We will describe a map by flags and monodromies.

$$M = (F; a, b, c)$$

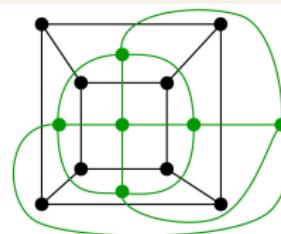
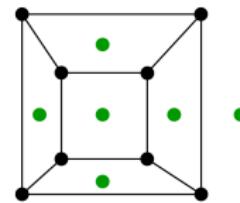
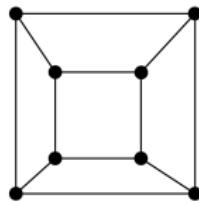


# Regular maps

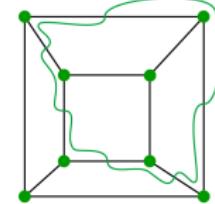
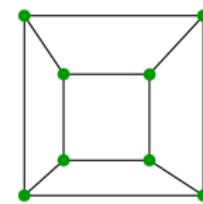
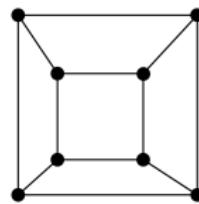
- The group generated by  $a$ ,  $b$ , and  $c$  is the group of monodromies.  $\text{Mon}(M) = \langle a, b, c \rangle$
- The monodromies do not preserve a map but there are permutations of flags which preserve a map. They are automorphisms. The group of all automorphisms will be denoted  $\text{Aut}(M)$ .
- A permutation  $\alpha$  is an automorphism of map iff  $\alpha$  commute with all monodromies.
- A map is regular if for each pair of flags  $(f_1, f_2)$  there is an automorphism  $\alpha$  such that  $\alpha(f_1) = f_2$ .
- In this case  $\text{Mon}(M) \cong \langle a, b, c \rangle \cong \text{Aut}(M)$ .
- A regular map can be identified with its automorphism group (more precisely, with its presentation in terms of the three generators above. We do not need  $F$  since it is equal to underlying set of  $\text{Aut}(M)$ .

# Map operators: Dual and Petri-Dual

Dual:  $M = (F; a, b, c) \rightarrow D(M) = (F; c, b, a)$



Petri-dual:  $M = (F; a, b, c) \rightarrow P(M) = (F; a, b, ac)$

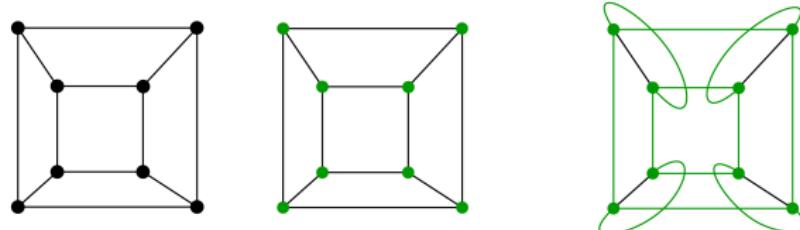


A map is self-dual if  $M = D(M)$ .

A map is self-Petri-dual if  $M = P(M)$ .

# Map operators: Hole operator $H_{-1}$ and $H_i$

Hole operator  $H_{-1}: M = (F; a, b, c) \rightarrow H_{-1}(M) = (F; a, aba, c)$



Hole operator  $H_i: M = (F; a, b, c) \rightarrow H_i(M) = (F; a, a(ab)^i, c)$

The parameter  $i$  must be coprime with the valence of all vertices.

The map  $M$  has the exponent  $i$  if  $M = H_i(M)$ .

Self-duality, self-Petri-duality and exponents are external symmetries of a map. All of them form the group of external symmetries  $E = \langle P, D, H_i \rangle$ .

All these external symmetries are automorphisms of  $\text{Aut}(M)$

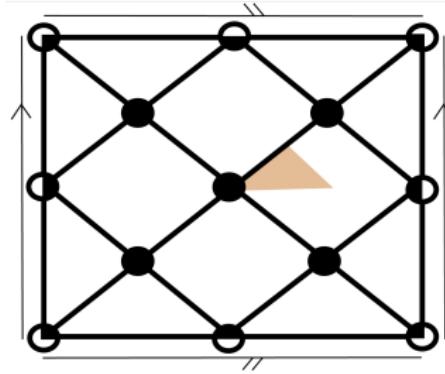
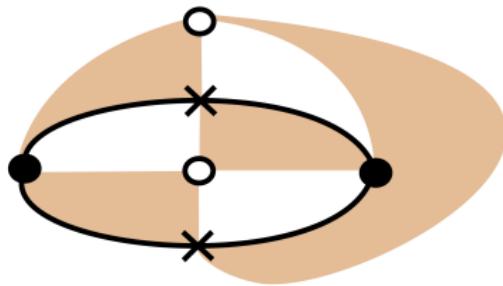


# One special case

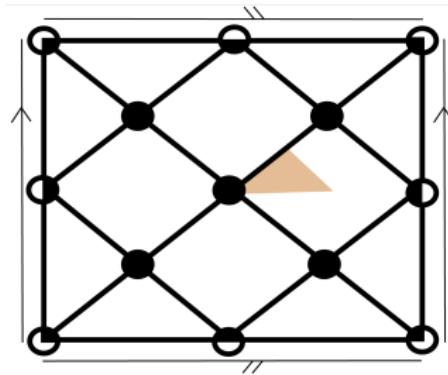
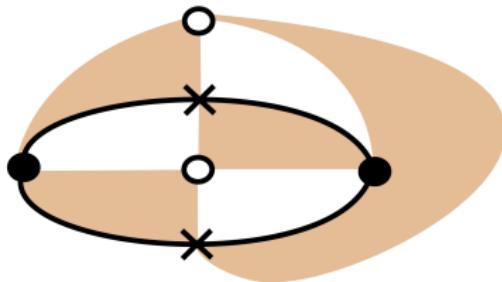
If a map is regular, self-dual, self-Petri-dual, and have all possible exponents, then it is the kaleidoscopic regular map with the trinity symmetry.

There is one infinity class of such maps given by their groups of automorphism  $G_k$  for each integer  $k \geq 1$ .

$$G_k = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^{2k} = (cb)^{2k} = (acb)^{2k} = (abacbc)^2 = 1 \rangle$$



# 'One' special case



$k = 1$ : Valency  $2k = 2$ ,

$|G_1| = 8$ ,  $|V| = 2$ ,  $|E| = 2$ ,  $|F| = 2$ ; map on sphere

$k = 2$ : Valency  $2k = 4$ ,

$|G_2| = 64$ ,  $|V| = 8$ ,  $|E| = 16$ ,  $|F| = 8$ ; map on torus

$k = 3$ : Valency  $2k = 6$ ,

$|G_3| = 216$ ,  $|V| = 18$ ,  $|E| = 54$ ,  $|F| = 18$ ; map on 11-torus.

I don't have a picture.

# Something missing

Denote by  $Aut^*(G_k)$  group of all automorphisms of  $G_k$  which fix  $\langle a, c \rangle$ .

It is known, that group of all external symmetries  $E(G_k)$  has size  $6(\varphi(2k))^3/2^\beta$  ( $\beta = d+2, d+1$  or  $d$ ;  $d$  number of prime factors of  $k$ ).

But it seems that  $|E| < |Aut^*(G_k)|$

# Something missing

$$E = F \rtimes S_3$$

$$F = \left\langle \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{array} \right), \left( \begin{array}{ccc} j & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & j \end{array} \right), \left( \begin{array}{ccc} l & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & 1 \end{array} \right) \right\rangle$$

where  $i, j, l \in Z_k^*$

$$Aut^*(G_k) = Aut'(G_k) \rtimes S_3$$

$$Aut'(G_k) = [k \text{ is odd}] = \left\langle \left( \begin{array}{ccc} i & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & l \end{array} \right) \right\rangle = (Z_k^*)^3 = (Z_{2k}^*)^3$$

$$[k \text{ is even}] = \left\langle \left( \begin{array}{ccc} i & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & l \end{array} \right) \right\rangle \times Z_2^3 = (Z_{2k}^*)^3$$

$$[k \text{ is eveeven}] = \left\langle \left( \begin{array}{ccc} i & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & l \end{array} \right) \right\rangle \Big|_{Z_2} \times Z_2^2 \times Z_4 = (Z_k^*)^3 \times Z_2 \times Z_4$$

# Extra external symmetries:

Dual  $D : a \rightarrow c, b \rightarrow b, c \rightarrow a$

Petri-dual  $P : a \rightarrow a, b \rightarrow b, c \rightarrow ac$

Hole operator  $H_i : a \rightarrow a, b \rightarrow a(ab)^i, c \rightarrow c$        $diag(1, e, e)$

$DH_iD : a \rightarrow a, b \rightarrow c(cb)^i, c \rightarrow c$        $diag(e, 1, e)$

$PDH_iDP : a \rightarrow a, b \rightarrow ac(acb)^i, c \rightarrow c$        $diag(e, e, 1)$

?1<sub>i</sub> :  $a \rightarrow a, b \rightarrow (bcbac)^{2i}b, c \rightarrow c$        $diag(e, 1, 1)$

?2<sub>i</sub> :  $a \rightarrow a, b \rightarrow (babac)^{2i}b, c \rightarrow c$        $diag(1, e, 1)$

?3<sub>i</sub> :  $a \rightarrow a, b \rightarrow (babc)^{2i}b, c \rightarrow c$        $diag(1, 1, e)$

## Acknowledgement

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Thank you for your attention, I look forward to your questions and remarks

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-  Dan Archdeacon, Marston Conder and Jozef Širáň, Trinity symmetry and kaleidoscopic regular maps, *Transactions of the American Mathematical Society* 366 (2014), 4491-4512.
-  Steve Wilson, New techniques for the construction of regular maps, doctoral dissertation, University of Washington, Seattle, 1976.
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# Myšlienka dôkazu:

- $G_k$  má komutatívnu podgrupu  $N_k = \langle u, v, w \rangle$ , kde  $u = abab$ ,  $v = cbc$  a  $w = acbacb$ .
- Na  $N_k$  sa automorfizmy  $G_k$  správajú ako transformácie 3-rozmerného vektorového priestoru a môžeme ich zapísať maticami.
- Tieto matice teraz chceme diagonalizovať, teda vhodne zmeniť bázu. Vďaka tomu, že všetky tieto matice navzajom komutujú, sa naozaj dajú naraz diagonalizovať.
- Ukážeme, že práve tie diagonálne matice, ktoré majú na diagonále len čísla nesúdeliteľné s  $k$ , reprezentujú automorfizmy  $G_k$ .